Analytical study of Dirac photon dispersion in metamaterials

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1. Introduction

Photon does not have a rest mass so that the relation between its energy ($E = \hbar\omega$) and momentum ($p = \hbar k$) is linear. However, in periodic structured materials like photonic crystals[1] and metamaterials,[2] the photon dispersion relation deviates quite a lot from its original linear relation due to modulation of eigenmodes induced by the periodic structures. In particular, the dispersion curves are horizontal in the vicinity of the Γ point of the Brillouin zone except the small ω limit where photon is propagated in an effectively uniform medium. We may regard that photon obtains an effective mass by interacting with the periodic structure.

On the other hand, realization of linear dispersion by accidental degeneracy of two dispersion curves of periodic metamaterial arrays has been extensively discussed by the transmission line theory.[2] In this presentation, I will reformulate this problem by the tight-binding approximation and group theory, and show that the photon effective mass induced by the periodic structure becomes vanishing again due to constraints imposed by the symmetry of the electromagnetic resonance states of the unit structure.[3,4] This phenomenon is a counter part of the Dirac fermion of graphene and might be called the Dirac photon.

2. Dirac photon in one-dimensional periodic metamaterials[3]

Since two bands are necessary to obtain the linear dispersion, I assume two electromagnetic resonance states for the single unit structure of the metamaterial and denote their magnetic field by $\mathbf{H}_0^{(j)}(\mathbf{r})$ (j = 1, 2). $\mathbf{H}_0^{(j)}$ satisfies the following eigen value equation:

$$\nabla \times \left[\frac{1}{\varepsilon_s(\mathbf{r})} \nabla \times \mathbf{H}_0^{(1,2)}(\mathbf{r})\right] = \frac{\omega_{1,2}^2}{c^2} \mathbf{H}_0^{(1,2)}(\mathbf{r}) \tag{1}$$

where $\varepsilon_s(\mathbf{r})$ is the dielectric constant of the system with one unit structure, which is assumed to have the C_{2v} symmetry in this section. For simplicity, $\varepsilon_s(\mathbf{r})$ is assumed to be real and the periodic boundary condition is applied so that ω_j is real. According to the tight-binding picture, the magnetic field of the one-dimensional periodic system should have the following form:

$$\mathbf{H}_k(\mathbf{r}) = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} e^{ikna} \left\{C_1 \mathbf{H}_0^{(1)}(\mathbf{r} - n\mathbf{a}) + C_2 \mathbf{H}_0^{(2)}(\mathbf{r} - n\mathbf{a})\right\}, \tag{2}$$

where $\mathbf{a}$ is the elementary lattice vector, a is the lattice constant, N is the number of unit structures on which the periodic boundary condition is imposed, and k is the wave number in the first Brillouin zone. C_1 and C_2 are constants that do not depend on n. Since $\mathbf{H}_k$ is an eigen function of the periodic system, it satisfies

$$\boldsymbol{\mathcal{L}} \mathbf{H}_\mathrm{k}(\mathbf{r}) \equiv \nabla \times \left[\frac{1}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}_\mathrm{k}(\mathbf{r})\right] = \frac{\omega_\mathrm{k}^2}{c^2} \mathbf{H}_\mathrm{k}(\mathbf{r}), \tag{3}$$

where $\varepsilon(\mathbf{r})$ is the dielectric constant of the periodic system. Then, the secular equations are derived according to the prescription of the tight-binding approximation. When the spatial symmetries of the two resonance states are A_1 and B_1, and the offset (B) of the two bands is adjusted to be equal to zero, then the next dispersion relation is obtained in the vicinity of the Γ point:

$$\omega_k = \omega_\Gamma \pm \frac{c^2 ka}{\omega_\Gamma} \left|L_1^{(12)}\right|, \tag{4}$$

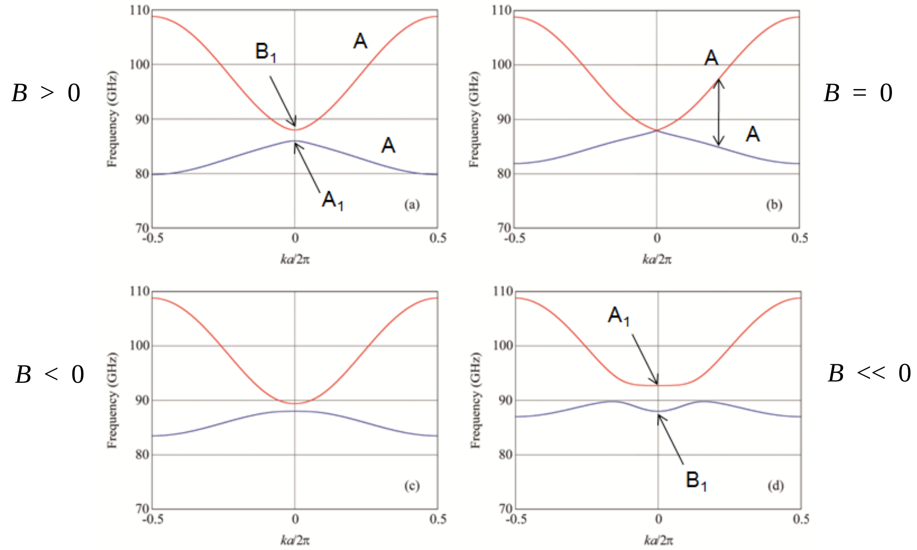

Fig. 1 Linear dispersion realized by the balance of crossing and anti-crossing

where ω_Γ is the angular frequency at the Γ point and L is the *electromagnetic transfer integral* that is defined as

$$L_n^{(ij)} = \frac{1}{V}\int_V d\mathbf{r}\, \mathbf{H}_0^{(i*)}(\mathbf{r}) \cdot \mathcal{L}\, \mathbf{H}_0^{(j)}(\mathbf{r} - n\mathbf{a}). \quad (5)$$

To examine the origin of this linear dispersion, the dispersion relation, which was obtained by solving the secular equation, is plotted for four B values in Fig. 1. Because the symmetries of the two bands on the Γ point, which are A_1 and B_1, are different from each other, the two bands cross each other. But for the rest of k in the first Brillouin zone, they have the same A symmetry, so that they repel each other. The balance of these crossing and anti-crossing behaviors brings about the linear dispersion relation around the Γ point when the offset $B = 0$ as shown in Fig. 1(b). We can easily prove that the linear dispersion can also be realized by the combination of the A_2 and B_2 resonance states of the C_{2v} point group.

3. Dirac photon in higher dimensions[4]

For two- and three-dimensional periodic arrays of unit structures, we have much more variations in combination of the resonance state symmetry and the lattice symmetry. In Ref. 4, degenerate resonance states of two-dimensional metamaterial arrays of the C_{4v} and C_{6v} symmetries were examined and it was shown that we can realize the linear dispersion by intentional symmetry reduction of the unit structure and fine adjustment of sample parameters. This technique that we call *controlled symmetry reduction* may be efficient in optical frequencies.

In addition to this example, I plan to describe more cases with different combinations of the mode and lattice symmetries.

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References

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