A circuit-theoretical method for analyzing coupled modes in metamaterials

Yosuke Nakata[1]*, Takanori Okada[2], Toshihiro Nakanishi[1], and Masao Kitano[1]
[1]*Department of Electronic Science and Engineering,*
Kyoto University, Kyoto 615-8510, Japan
[2]*Pioneering Research Unit for Next Generation,*
Kyoto University, Kyoto 615-8245, Japan
**E-mail:nakata@giga.kuee.kyoto-u.ac.jp*

Metamaterials are engineered composite electromagnetic media consisting of lower-level components, or *meta atoms* (Ref. 1). In most of metamaterials, metallic structures behaving as resonators are utilized for meta atoms. Due to the resonances metamaterials respond strongly and unusually for electromagnetic waves. Such meta atoms can be modeled as LC resonant circuits. At early stages of metamaterial research, meta atoms much smaller than the working wavelength were used and the coupling between them are ignored. On this condition, each meta atom responds independently for electromagnetic waves. Such metamaterials behave as effective media described by macroscopic parameters such as electrical permittivity and magnetic permeability independent of wave vectors. They exhibit averaged effects of the resonance property of the individual meta atoms. By engineering the electrical permittivity and magnetic permeability, unusual electromagnetic responses that are not observed in natural materials can be obtained, such as a negative refractive index, artificial magnetism, and super focusing, cloaking.

Although such various inventions can be achieved, their properties were often based only on averaged effects of meta atoms, and coupling effects were ignored. If there exist couplings between meta atoms, hybridization effects such as multiple modes and continuum bands must be considered. Recently, these hybridization effects in metamaterials have attracted great interest (for review, see Ref. 2).

However, there has been no general theory to analyze the hybridization modes in metamaterials. Therefore, it is needed to construct a theoretical framework to analyze them. For this purpose, a circuit model of metamaterials is introduced. If the metamaterials have only small dissipation, we can ignore the resistance of the circuits. It is enough to consider the circuit networks composed of (linear) inductors and capacitors. Here, inductors can interact with each other by mutual inductance. Similarly, capacitors can interact by mutual capacitance. Such networks are called inductor-capacitor circuit networks (Fig. 1).

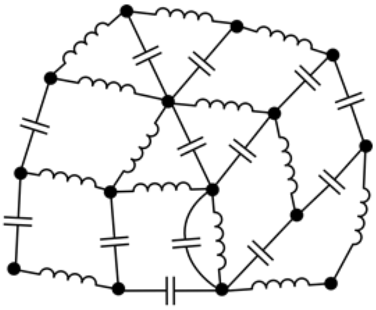

FIG 1. An example of inductor-capacitor networks

In this research, we formulate a general method to analyze resonances in an arbitrary inductor-capacitor network. For this purpose, we treat them in a frequency domain and all variables are described by phasors (complex amplitudes). Here, the resonant condition is that there exists nontrivial (nonzero) distribution of current or voltage which satisfies Kirchhoff's current law (KCL), Kirchhoff's voltage law (KVL) and the all device characteristics. In order to write down these conditions, there are two possible choices of basic variables. The first choice is selecting the mesh currents satisfying KCL. Second one is using the electrical potentials satisfying KVL. We discuss the both methods, adapting bra-ket notation. By using this notation, we can perform the calculation like in quantum mechanics familiar to physicists.

Next, in order to actually show that this analogy is not only notational but also physical one, we apply our theoretical method to two specific inductor-capacitor circuit networks shown in Fig. 2. For these models, we can formulate the resonance conditions like the tight-binding models in quantum mechanics. These *quantum-mechanical analogies for electrical circuits* enable us to realize the dispersion similar to well-known quantum tight-binding model in metamaterials.

Next, to demonstrate the usefulness of this analogy, we predict new phenomena in metamaterials. For this, we take into account the emergence of electron flat dispersion relations independent of wave vectors in the solid states with some symmetries. At such flat bands, the group velocity of electrons is slowed down for all wave vectors, and the mass becomes very heavy. Then, such an electron is called *a heavy electron*. For the heavy electron, the electron correlation exceeds kinetic energies and ferromagnetism can occur. Actually, it is theoretically known that Lieb-type (Ref. 3), Tasaki-type (Ref. 4), and Mielke-type (Ref. 5) symmetries are known to cause flat bands and ferromagnetism. Therefore, they are drawing great attention in physics. We consider analogies of heavy electrons for inductor-capacitor circuit networks, namely, *heavy photons* (strictly speaking, *heavy plasmon polaritons*). With these three lattices' symmetry, we theoretically showed that these flat bands are generally formed. For concrete examples, we calculate band structures for the three network symmetries and obtain electromagnetic flat bands (Fig. 3).

This result shows the prediction power of our theoretical frameworks. Of course, other band structures in quantum tight-binding models can be realized in inductor-capacitor networks. Then, our theoretical method leads to new dispersion design method in metamaterials by using quantum-mechanical analogy for electrical circuits. This is a promising approach to design hybridization modes in metamaterials.

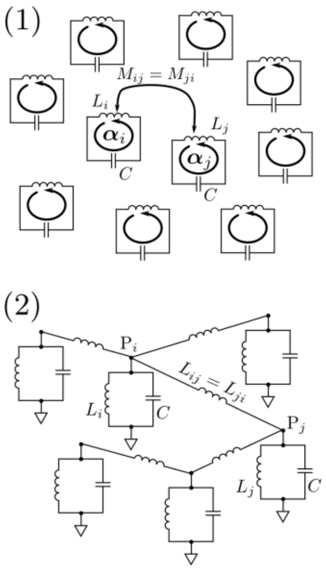

FIG 2. Inductor-capacitor networks corresponding to quantum tight-binding

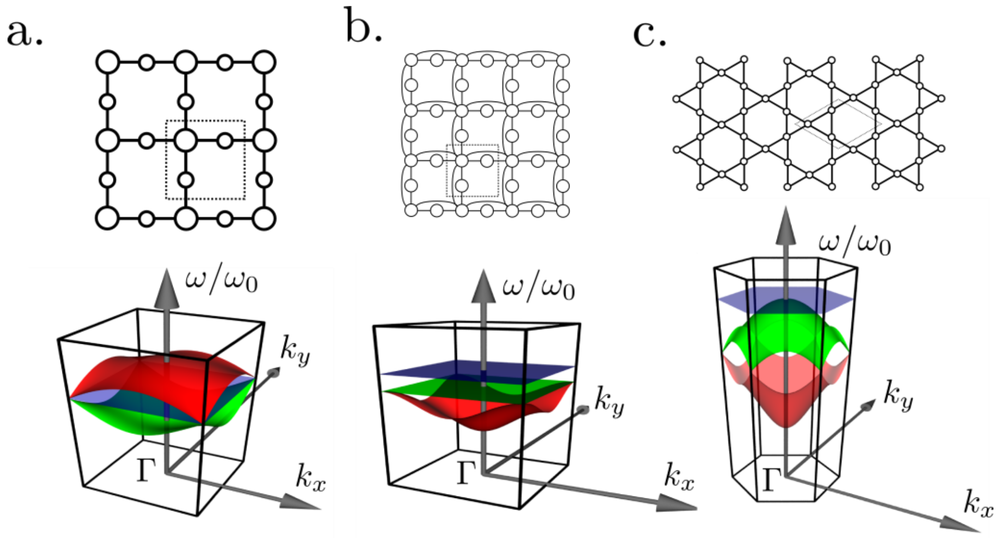

FIG 3. Three symmetries causing flat bands and actual band structures in the inductor-capacitor circuits corresponding to upper symmetries. For concrete calculation, we use the resonant circuits coupled by choke coils shown in Fig. 2(2).

References

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